Europ. J. Combinatorics (1993) **14**, 59–77

On Some Flag-transitive Non-classical $c.C_2$-geometries

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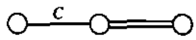
Construction and characterization is given for three new flag-transitive non-classical extended generalized quadrangles. They are simply connected with point-residues the non-classical generalized quadrangle $T_2^*(O_4)$ and its dual $T_2^{**}(O_4)$.

1. INTRODUCTION

A *$c.C_2$-geometry* is a residually connected incidence geometry $\mathscr{G} = (\mathscr{G}_0, \mathscr{G}_1, \mathscr{G}_2; *)$ over $I = \{0, 1, 2\}$ belonging to the following diagram:



that is, the residues at elements of $\mathscr{G}_0$, $\mathscr{G}_1$ and $\mathscr{G}_2$ are isomorphic to a generalized quadrangle (abbreviated to GQ), a generalized digon and a circle geometry, respectively. We call elements of $\mathscr{G}_0$, $\mathscr{G}_1$ and $\mathscr{G}_2$ *points, lines* and *planes*, respectively. As usual, we denote by $\mathscr{G}_i(x)$ the set of elements of $\mathscr{G}_i$ incident with an element $x \in \bigcup_{j=0}^{2} \mathscr{G}_j$.

For convenience, in this paper we will abbreviate a flag-transitive $c.C_2$-geometry to an *FEQ*. Note that this does *not* mean that a flag-transitive $c.C_2$-geometry is always an extended generalized quadrangle in the sense of [3]; that is, a $c.C_2$-geometry satisfying the intersection property (see [3, p. 194; 4, p. 255]).

Since the point residues of an FEQ are isomorphic to a flag-transitive GQ, it would be desirable to classify them. Unfortunately, such a classification has not yet been completed. At the present time, the following thick GQs are known to be flag-transitive (see [6, p. 98, summary]):

(c) Six infinite families of GQs, called *classical* GQs: $W(q)$, $Q(4, q)$, $Q^-(5, q)$, $H(3, q^2)$, $H(4, q^2)$ and its dual for any prime power q (see [8, 3.1.1, p. 36] for the notation).

(e) Two GQs $T_2^*(O_q)$ for some oval O_q in the projective plane $\mathbf{PG}(2, q)$ for $q = 4$ and 16 and their duals $T_2^{**}(O_q)$ (see [8, 3.1.3, p. 38]).

An FEQ is called *classical* if its point-residues are isomorphic to one of the thick classical GQs above.

By [1, 4] and [9], classical FEQs are completely classified. (Note that FEQs with point-residues the dual of $H(4, q^2)$ are not treated explicitly in the above literature. However, for example, by the argument used in Lemma 12 in [10], it is easy to verify that there is no such FEQ.) They turn out to satisfy the following property (LL):

(LL) For two distinct points, there is at most one line through them.

That is, the point–line graph of $\mathscr{G}$ does not contain multiple edges.

How about the non-classical FEQs? The main aim of this paper is to show the existence of four new FEQs with point-residues $T_2^*(O_4)$ and its dual $T_2^{**}(O_4)$ satisfying the (LL) property, together with their characterization. (Note that the full automorphism group of $T_2^*(O_4)$ is isomorphic to $2^6 3S_6$, in which 2^6 acts regularly on points: see 2.2.)

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THEOREM 1.1. *Up to isomorphism, there is a unique simply connected FEQ with point-residues isomorphic to $T_2^*(O_4)$, satisfying the (LL) property and admitting an automorphism group G in which the stabilizer of a point P contains a normal subgroup inducing a regular permutation group on the lines incident with P.*

This new FEQ $\mathcal{G}$ is defined on 160 points, having 3072 planes and the full automorphism group $2_+^{1+8}\cdot(A_5\times A_5)2$ (the extension $\mathrm{Aut}(\mathcal{G})/2_+^{1+8}$ does not split). Taking the quotient by the unique central involution of $\mathrm{Aut}(\mathcal{G})$, we have an FEQ on 80 points. So far, the only construction of this geometry known to the author is one in terms of coset geometry. For the details, see 3.8.

THEOREM 1.2. *Up to isomorphism, there are two simply connected FEQs with point residues isomorphic to the dual of $T_2^*(O_4)$, satisfying the (LL) property and admitting an automorphism group G in which the stabilizer of a point P contains a normal subgroup inducing a regular permutation group on the lines incident with P.*

One of these new FEQs is $\mathcal{G}^{(0)}$ (resp. $\mathcal{G}^{(1)}$) defined on 896 (resp. 448) points, having 8192 (resp. 4096) planes and the full automorphism group $2_+^{1+12}{:}\,3S_7$ (resp. $2^{6+6}{:}\,L_3(2)$). So far, the only construction of the geometry $\mathcal{G}^{(1)}$ known to the author is one in terms of coset geometry. For the details, see 4.2. An explicit construction of $\mathcal{G}^{(0)}$ will be given in 4.3 in terms of isotropic 1-, 2- and 4-spaces of an 8-dimensional unitary space over $\mathbf{F}_4$. Taking the quotient by the unique central involution of $\mathrm{Aut}(\mathcal{G}^{(0)})$, we have an FEQ on 448 points.

Using detailed information about the automorphism group of $T_2^*(O_4)$ prepared in Section 2, we will prove Theorem 1.1 and 1.2 in Sections 3 and 4, respectively, together with some information about the resulting new geometries. The method adopted there is based on generators and relations, which were described in [10, Sections 1, 2]. Since there are many flag-transitive subgroups of $\mathrm{Aut}(T_2^*(O_4))$, the classification requires similar considerations for many cases. I give the detailed arguments only for a few cases, but sketch out the others. Some familiarity with fundamental definitions [7] and the arguments in [10, Sections 1, 2] is assumed.

2. GQ T_2^* (O_4) AND ITS AUTOMORPHISM GROUP

2.1. *The GQ $T_2^*(O)$*

We first review a construction of $T_2^*(O)$ (see [8, 3.1.3]). Let $\mathbf{PG}(n, q)$ be a projective space of dimension n of order a 2-power q, consisting of projective points $[x_0, \ldots, x_n]$ determined by vectors $(x_0, \ldots, x_n)\in \mathbf{F}_q^{n+1}$. We take a hyperplane H of $\mathbf{PG}(3, q)$, identified with a projective plane $\mathbf{PG}(2, q)$, and fix any *oval* O of H: O is a set of $(q+2)$ points of H, in which no distinct three points lie on a line in common. It is known that any ovals in $\mathbf{PG}(2, 4)$ are equivalent to each other, and that there are two equivalence classes of ovals in $\mathbf{PG}(2, 16)$ (see [5, p. 177]). For example, the set $O_4 := \{[1, 0, 0],\ [0, 1, 0],\ [0, 0, 1],\ [1, 1, 1],\ [1, \omega, \omega^2],\ [1, \omega^2, \omega]\} = \{[1, 0, 0], [x, y, z] \mid x, y, z \in \mathbf{F}_4, x^2 + yz = 0\}$ is an oval in $\mathbf{PG}(2, 4)$, where ω is a primitive element of $\mathbf{F}_4$. As representatives of the two equivalence classes of ovals in $\mathbf{PG}(2, 16)$, we may take $O_{16} := \{[1, 0, 0],\ [0, 1, 0],\ [F(t), t, 1] \mid t \in \mathbf{F}_{16}\}$ and $O'_{16} := \{[1, 0, 0], [x, y, z] \mid x, y, z \in \mathbf{F}_{16}, x^2 + yz = 0\}$, where $F(x) = (\eta^2x^7 + \eta^{12}x^6 + \eta^6x^5 + \eta^9x^4 + \eta^5x^3 + \eta^5x^2 + \eta^6x)^2$ and η is a primitive element of $\mathbf{F}_{16}$ with $\eta^4 = \eta + 1$. We construct a generalized quadrangle $T_2^*(O)$ as follows.

Let Γ_0 be the set of points of $\mathbf{PG}(3,q)$ not contained in H, and let Γ_1 be the set of lines of $\mathbf{PG}(3,q)$ intersecting H at a point of O. Incidence is determined by inclusion. Then we may verify that the resulting geometry $\mathcal{G}=(\mathcal{G}_0,\mathcal{G}_1;*)$ is a GQ of order $(q-1,q+1)$.

By the above construction of $T_2^*(O)$, we can easily find a subgroup of $\mathrm{Aut}(T_2^*(O))$ inside $PGL_4(q)$. In the group $PGL_4(q)$ acting faithfully on $\mathbf{PG}(3,q)$, the stabilizer of a hyperplane $H=\{[0,x,y,z]\mid x,y,z\in\mathbf{F}_q\}$ is $P/\langle\omega I_4\rangle$, where ω is a primitive element of $\mathbf{F}_q$, I_4 denotes the identity matrix of degree 4 and the group P consists of matrices

$$M(\alpha,\mathbf{a},X):=\begin{pmatrix}\alpha & \mathbf{a}\\ \mathbf{0}^t & X\end{pmatrix}$$

for $\alpha\in\mathbf{F}_q^\times$, $\mathbf{a}\in\mathbf{F}_q^3$ and $X\in GL_3(q)$. We also set $L:=\{M(1,\mathbf{0},X)\mid X\in GL_3(q)\}$, $N:=\{M(1,\mathbf{a},I_3)\mid\mathbf{a}\in\mathbf{F}_q^3\}$ and $d:=M(1,\mathbf{0},\omega I_3)$. Then $P/\langle\omega I_4\rangle\simeq NL\simeq 2^{3e}GL_3(q)$, in which $N\langle d\rangle$ is the kernel of the action of P on H, where $q=2^e$.

Now let A be the stabilizer of O in $L\simeq GL_3(q)$. Then the subgroup $\langle\omega I_4\rangle NA/\langle\omega I_4\rangle\simeq NA$ of $P/\langle\omega I_4\rangle$ acts faithfully on $\mathbf{PG}(3,q)$ stabilizing H and O. Thus, by the above construction of the GQ $T_2^*(O)$, NA is an automorphism group of $T_2^*(O)$. Furthermore, if O is stabilized by a field automorphism σ of $\mathbf{PG}(3,q)$, σ also induces an automorphism of $T_2^*(O)$.

2.2. *Lemma*

The full automorphism group X of $\Gamma:=T_2^*(O_4)$ is isomorphic to $2^6{:}\,(3A_6)2$. For a flag (α,β), the stabilizer X_α of a point α is isomorphic to $(3A_6)2$ with kernel 3, while the stabilizer X_β of a line β is isomorphic to $(A_5\times A_4)2$ with kernel $(X_{\alpha,\beta})^\infty\simeq A_5$. The groups X_α and X_β induces S_6 and S_4 on the set $\Gamma_1(\alpha)$ of six lines through α and the set $\Gamma_0(\beta)$ of four points on β, respectively. Moreover, the group fixing a point (resp. lines) and all points collinear with it (resp. all lines intersecting it) is trivial.

PROOF. We denote by Q_i $(i=1,\dots,6)$ the six points $[0,1,0,0]$, $[0,0,1,0]$, $[0,0,0,1]$, $[0,1,1,1]$, $[0,1,\omega,\omega^2]$, $[0,1,\omega^2,\omega]$ of the hyperplane $H=\{[0,x,y,z]\mid x,y,z\in\mathbf{F}_4\}$ respectively. The set $O_4:=\{Q_i\mid i=1,\dots,6\}$ is an oval on H stabilized by the field automorphism σ given by $x^\sigma=x^2(x\in\mathbf{F}_4)$. It is shown that the stabilizer in $PGL_3(4)$ of O_4 is isomorphic to A_6 ([5, p. 178, Cor. 6]). This stabilizer lies in $PSL_3(4)$ of index 56. Since this index is prime to 3 and $SL_3(4)$ is a non-split central extension of $PSL_3(4)$, the stabilizer A of O_4 in $GL_3(4)$ is isomorphic to the non-split extension $3A_6$. Thus it follows from the above argument that there is a subgroup $NA\langle\sigma\rangle\simeq 2^6{:}\,3S_6$ of $X:=\mathrm{Aut}(\Gamma)$.

For explicitness, we list generators of $A\langle\sigma\rangle\simeq 3S_6$. Set $s_i:=M(1,0,g_i)\sigma$ $(i=1,\dots,4)$ and $s_5:=\sigma$, where g_i are the following matrices of $GL_3(4)$:

$$g_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\quad g_2=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad g_3=\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix},\quad g_4=\begin{pmatrix}1&0&0\\0&\omega^2&0\\0&0&\omega\end{pmatrix}.$$

We may verify that d and s_i satisfy the following presentation for $3S_6$, in which $(3S_6)'$ is a non-split triple central extension of A_6. The element s_i induces the transposition (Q_i,Q_{i+1}) on the six points of O_4:

$$\mathcal{R}{:}\ d^3=1,\quad s_i^2=1,\quad d^{s_i}=d^{-1}\ (i=1,\dots,5),\quad (s_is_{i+1})^3=1\ (i=1,\dots,4),$$
$$[s_1,s_3]=1,\quad (s_1s_4)^2=d\quad\text{and}\quad [s_1,s_5]=[s_2,s_4]=[s_2,s_5]=[s_3,s_5]=1.$$

We now recall the proof of the characterization of the GQ $\Gamma := T_2^*(O_4)$ as a GQ of order $(3,5)$ by Dixmier and Zara ([8], pp. 125–129, especially steps (g) and (h)). There are 96 lines of Γ, which are divided into six parallel classes corresponding to the six points of O_4. Since $A\langle\sigma\rangle$ induces S_6 on these points, we have $\mathrm{Aut}(\Gamma)/K \simeq S_6$, where K is the kernel of the action of $\mathrm{Aut}(\Gamma)$ on six parallel classes. Using these classes E, E_i $(i = 1, \ldots, 5)$, a new GQ $\mathscr{S}'$ can be constructed from Γ as in the step (g) of the above-mentioned proof ([8, p. 128]). Since K fixes each parallel class, it induces an automorphism group of the GQ $\mathscr{S}'$ fixing the new point ∞ and any five lines E_i through ∞ (see [8]). By the step (h) of the proof ([8], p. 128]), the GQ $\mathscr{S}$ is isomorphic to the GQ $W(4)$ with $\mathrm{Aut}(W(4)) \simeq S_4(4).2$. In $\mathrm{Aut}(W(4))$, the stabilizer of a point fixing five lines through it is isomorphic to $2^6 3$. Thus, $|\mathrm{Aut}(\Gamma)| \leqslant |2^6 3 S_6|$ and $\mathrm{Aut}(\Gamma)$ was determined.

The stabilizer X_α of a point $\alpha := [1, 0, 0, 0]$ in $X := \mathrm{Aut}(\Gamma)$ coincides with $A\langle\sigma\rangle \simeq 3S_6$. Since the six lines of $\Gamma_1(\alpha)$ correspond to the six points on the oval O_4, the kernel of X_α on $\Gamma_1(\alpha)$ is $K_\alpha = \langle d\rangle$. The group fixing the point α and all points collinear with α is trivial, since d cyclically permutes three points except α on any line through α.

The six lines through the point α correspond to six points Q_i $(i = 1, \ldots, 6)$ on O_4. Let β be the line through α and Q_1. Since the stabilizer $X_{\alpha,\beta}$ consists of elements of X_α fixing Q_1, it is isomorphic to $(3 \times A_5)2$. Thus $X_{\alpha,\beta} = (\langle d\rangle \times B)\langle s_5\rangle$, where $B \simeq A_5$ is generated by $a_{5-i} := s_i s_5$ $(i = 2, 3, 4)$, inducing permutations $(Q_i, Q_{i+1})(Q_4, Q_5)$, respectively. By the relations $\mathscr{R}$, they satisfy a presentation for A_5: $a_1^3 = a_2^2 = a_3^2 = (a_1a_3)^3 = (a_1a_3)^2 = 1$. Since there are three points distinct from α on the line β, the group B coincides with the kernel K_β of the action of the stabilizer X_β on $\Gamma_0(\beta)$. The group fixing β and all lines intersecting β is a subgroup of B stabilizing all points on O_4, and therefore it is trivial. We set $e_1 := M(1, \mathbf{e}_1, I_3)$, where $\mathbf{e}_1$ denotes $(1,0,0)$. Then the group $\langle e_1, d, \sigma\rangle \simeq S_4$ acts faithfully on $\Gamma_0(\beta)$, and therefore we have $X_\beta = (B \times \langle e_1, d\rangle)\langle\sigma\rangle \simeq (A_5 \times A_4)2$. In particular, we have $B = K_\beta = C_{X_{\alpha,\beta}}(e_1, d)$. □

2.3. *Some flag-transitive subgroups of* $\mathrm{Aut}(T_2^*(O_4))$

We now consider a flag-transitive subgroups of $X := \mathrm{Aut}(T_2^*(O_4))$. However, it seems that there are too many such subgroups. Thus, in this paper, we only consider these subgroups containing a normal subgroup $N := O_2(X) \simeq 2^6$, acting regularly on the points of Γ.

Let Y be a subgroup of X acting flag-transitively on $\Gamma := T_2^*(O_4)$. There are six parallel classes of lines of Γ, each of which consists of lines of $\mathbf{PG}(3,4)$ through a point of the oval O_4. The group X induces S_6 with kernel $K = N\langle d\rangle \simeq 2^6 3$ on the set $\mathscr{P}$ of six parallel classes. For short, we denote by α and β the point $[1,0,0,0]$ and the line $[\langle\alpha, Q_1\rangle] = \{[x, y, 0, 0 \mid x, y \in \mathbf{F}_4\}$ of $\Gamma_1(\alpha)$, respectively. We identify the letter i $(i = 1, \ldots, 6)$ with the representative line l of $\mathscr{P}$ through α and Q_i, and in turn with the parallel class containing l. Then X/K and $X_\alpha/\langle d\rangle$ can be identified with the symmetric group S_6.

Since X is a split extension of X_α by N, we have $Y = NY_\alpha$ by our assumption $N \subseteq Y$. Since six lines of $\Gamma_1(\alpha)$ form a complete representative of $\mathscr{P}$, Y acts flag-transitivity on Γ iff the stabilizer Y_α acts transitively on $\mathscr{P}$. In order to determine a subgroup $Y_\alpha/Y_\alpha \cap \langle d\rangle$ of $X/K \simeq S_6$, we need the following lemma, which can be verified by a standard group-theoretic argument (see also [2, p. 870, 874]).

2.4. *Lemma*

Let S be the symmetric group of degree 6 acting naturally on the set $\mathscr{P} = \{1, \ldots, 6\}$ of six letters. Then there are 16 conjugacy classes of subgroups of S acting transitively on $\mathscr{P}$

with the following representatives F_i ($i = 1, \ldots, 16$). In the 4th, 5th, 6th columns of the table below, we also include the information whether F_i contains a 2-element inducing an odd permutation on $\mathcal{P}$ (type 2^3, 2^1 and 4^1, respectively). The last column shows the stabilizer of a letter in $F_i \cap A_6$.

Name	*Iso. type*	*Generators*	2^3	2^1	4^1	*Stab.*
F_1	S_6	(12), (23), (34), (45), (56)	Y	Y	Y	A_5
F_2	A_6	(123), (12)(34), (12)(45), (12)(56)	N	N	N	A_5
F_3	S_5	u, p, q, m	Y	N	Y	F_5^2
F_4	A_5	$u, p, q = (13)(45)$	N	N	N	F_5^2
F_5	$S_4 \times 2$	u, p, l, m	Y	Y	Y	E_4
F_6	S_4	$u, p, n = (23)(56)$	N	N	N	E_4
F_7	$A_4 \times 2$	u, p, l	Y	Y	N	2
F_8	S_4	u, p, m	Y	N	Y	2
F_9	A_4	$u, p = (25)(36)$	N	N	N	2
F_{10}	3^2D_8	a, b, f, (12)	Y	Y	N	S_3
F_{11}	3^24	$a, b, f = (2356)(14)$	N	N	N	S_3
F_{12}	3^22^2	a, b, m, l	Y	N	N	S_3
F_{13}	3^22	$a = (123), b = (456), l$	Y	N	N	3
F_{14}	$S_3 \times 2$	u, m, l	Y	N	N	1
F_{15}	S_3	$u, m = (14)(26)(35)$	Y	N	N	1
F_{16}	6	$u = (123)(456), l = (14)(25)(36)$	Y	N	N	1

Furthermore, up to conjugacy, the following inclusion relations hold among the representatives above. In particular, $\{F_2, F_3, F_5, F_{10}\}$ (resp. $\{F_{16}, F_{15}, F_{11}, F_9\}$) is the set of representatives of conjugacy classes of maximal (resp. minimal) subgroups of S acting transitively on $\mathcal{P}$.

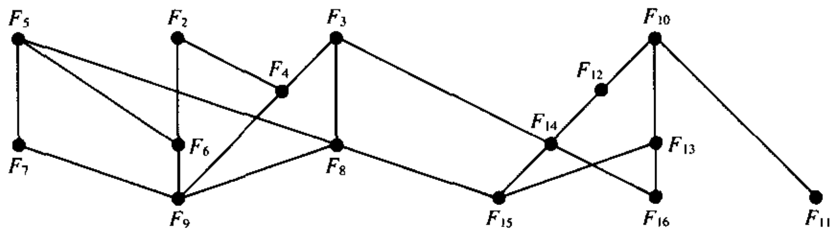

FIGURE 1. Inclusions of subgroups F_i.

2.5. *Presentations for some subgroups of* $\mathrm{Aut}(T_2^*(O_4))$

For the proofs of Theorem 1.1 and 1.2 in Sections 3 and 4, it is necessary to have explicit presentations for some flag-transitive subgroups of $X = \mathrm{Aut}(T_2^*(O_4))$. By 2.3, a flag-transitive subgroup Y of X is a semidirect product of N by Y_α. Thus in order to give a presentation of Y, it suffices to determine the generators and relations of Y_α and their actions on N. In this subsection, we consider Y in which $\langle d \rangle \subseteq Y_\alpha$ and $Y_\alpha / \langle d \rangle$ is a transitive subgroup F_i in the table of 2.4, for $i = 1, 3, 6, 7, 8, 9, 11, 14, 15$ or 16.

We will follow the notation in 2.1,2. We also identify the subgroup Y_α with a subgroup of $A\langle \sigma \rangle \simeq 3S_6$ in 2.2, and denote the element $M(1, 0, g)\sigma^i$ by $g\sigma^i$. Now we set elements a, b, $u := d^{-1}ab$, p, q, l, m, $n := g_2$ (see 2.2) and f of X_α as follows:

$$a=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\quad b=\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\bar\omega\end{pmatrix},\quad u=\begin{pmatrix}0&1&0\\0&0&\omega\\\bar\omega&0&0\end{pmatrix},\quad p=\begin{pmatrix}1&0&0\\1&\omega&\bar\omega\\1&\bar\omega&\omega\end{pmatrix},$$

$$q=\begin{pmatrix}0&0&\bar\omega\\0&1&0\\\bar\omega&0&0\end{pmatrix},\quad l=\begin{pmatrix}\bar\omega&\bar\omega&\bar\omega\\\bar\omega&\omega&1\\\bar\omega&1&\omega\end{pmatrix}\cdot\sigma,\quad m=\begin{pmatrix}\omega&\omega&\omega\\\omega&\bar\omega&1\\\omega&1&\bar\omega\end{pmatrix}\cdot\sigma,\quad f=\begin{pmatrix}1&1&1\\1&\omega&\bar\omega\\1&\bar\omega&\omega\end{pmatrix}.$$

We denote by $\bar g$ the permutation induced by g on $\mathscr{G}_2(l)$. Via the identification $\Gamma_1(\alpha)$ with six points Q_i $(i = 1, \ldots, 6)$ on the oval O_4 (see 2.3), we may verify that $\bar a = (123)$, $\bar b = (456)$, $\bar u = (123)(456)$, $\bar p = (25)(36)$, $\bar q = (13)(45)$, $\bar l = (14)(25)(36)$, $\bar m = (14)(26)(35)$, $\bar n = (23)(56)$ and $\bar f = (2356)(14)$. Furthermore, $\bar s_i = (i, i+1)$ for $i = 1, \ldots, 5$ (see 2.2). Thus it follows from the table in 2.4 that $\langle s_i \mid i = 1, \ldots, 5\rangle/\langle d\rangle$ $(\simeq S_6)$, $\langle d, u, p, q, m\rangle/\langle d\rangle$ $(\simeq S_5)$, $\langle d, u, p, lm\rangle/\langle d\rangle$ $(\simeq S_4)$, $\langle d, u, p, l\rangle/\langle d\rangle$ $(\simeq A_4 \times 2)$, $\langle d, u, p, m\rangle/\langle d\rangle$ $(\simeq S_4)$, $\langle d, u, p\rangle/\langle d\rangle$ $(\simeq A_4)$, $\langle d, a, b, f\rangle/\langle d\rangle$ $(\simeq 3^2 4)$, $\langle d, u, m\rangle/\langle d\rangle$ $(\simeq S_3)$ and $\langle d, u, l\rangle/\langle d\rangle$ $(\simeq 6)$ coincide with the subgroups F_i of S_6 for $i = 1, 3, 6, 7, 8, 9, 11, 15$ and 16, respectively.

The regular normal subgroup N on points is generated by $e_i = M(1, \mathbf{e}_i, I_3)$ $(i = 1, 2, 3)$ and their conjugates by d, where $\mathbf{e}_i$ denotes the ith natural basis of $\mathbf{F}_4^3$ $(i = 1, 2, 3)$. Note that the action of d corresponds to the multiplication by ω. They satisfy the following relations $\mathscr{E}$, which give a presentation of N:

$$\mathscr{E}\colon e_i^2 = [e_i, e_j] = [e_i, e_j^d] = e_i e_i^d e_i^{d^{-1}} = 1 \qquad \text{for any } 1 \leqslant i,\quad j \leqslant 3.$$

Since $M(1, \mathbf{0}, X)^{-1} e_i M(1, \mathbf{0}, X) = M(1, \mathbf{e}_i X, I_3)$ $(i = 1, 2, 3)$, we can immediately obtain the action of elements of X_α on N from their shapes as matrices:

$$\mathscr{F}\colon e_1^{s_1} = e_2,\quad e_3^{s_1} = e_3,\quad e_1^{s_2} = e_1,\quad e_2^{s_2} = e_3,\quad e_1^{s_3} = e_1,\quad e_2^{s_3} = e_2,\quad e_3^{s_3} = e_1e_2e_3,$$
$$e_1^{s_4} = e_1,\quad e_2^{s_4} = e_2^d,\quad e_3^{s_4} = e_3^{d^{-1}},\quad [e_1, s_5] = [e_2, s_5] = [e_3, s_5] = 1.$$
$$\mathscr{F}(a)\colon e_1^a = e_2,\quad e_2^a = e_3,\quad e_3^a = e_1;$$
$$\mathscr{F}(b)\colon e_1^b = e_1,\quad e_2^b = e_2^d,\quad e_3^b = e_3^{d^{-1}};$$
$$\mathscr{F}(u)\colon e_1^u = e_2,\quad e_2^u = e_3^d,\quad e_3^u = e_1^{d^{-1}};$$
$$\mathscr{F}(l)\colon e_1^l = e_1^d e_2^d e_3^d,\quad e_2^l = e_1^d e_2^{d^{-1}} e_3,\quad e_3^l = e_1^d e_2 e_3^{d^{-1}};$$
$$\mathscr{F}(n)\colon e_1^m = e_1^{d^{-1}} e_2^{d^{-1}} e_3^{d^{-1}},\quad e_2^m = e_1^{d^{-1}} e_2^d e_3,\quad e_3^m = e_1^{d^{-1}} e_2 e_3^d;$$
$$\mathscr{F}(f)\colon e_1^f = e_1e_2e_3,\quad e_2^f = e_1 e_2^d e_3^{d^{-1}},\quad e_3^f = e_1 e_2^{d^{-1}} e_3^d;$$
$$\mathscr{F}(n)\colon e_1^n = e_1,\quad e_2^n = e_3,\quad e_3^n = e_2;$$
$$\mathscr{F}(p)\colon e_1^p = e_1,\quad e_2^p = e_1 e_2^d e_3^{d^{-1}},\quad e_3^p = e_1 e_2^{d^{-1}} e_3^d.$$

Thus we may verify that the following set $\mathscr{R}_i$ of relations gives a presentation for a flag-transitive subgroup Y of X, in which $\langle d\rangle \subseteq Y_\alpha$ and $Y_\alpha/\langle d\rangle$ if F_i $(i = 1, 3, 6, 7, 8, 9, 11, 14, 15, 16)$. In the list below, $j = 1, 2, 3$ and $k = 1, \ldots, 5$:

Generators		*Relations*
e_j, d, s_k	$\mathscr{R}_1$	$\mathscr{E}$, $\mathscr{R}$ in 2.2 and $\mathscr{F}$.
e_j, d, u, p, q, m	$\mathscr{R}_3$	$\mathscr{E}$; $\mathscr{F}(g)$ for $g = u, p, q, m$, $d^3 = 1$; $u^3 = p^2 = q^2 = m^2 = 1$, $[d, u] = [d, p] = [d, q] = 1$, $\quad d^m = d^{-1}$, $(up)^3 = (pq)^3 = 1$, $(uq)^2 = (um)^2 = (pm)^2 = (qm)^2 = 1$.

e_j, d, u, p, n	$\mathcal{R}_6$	$\mathscr{E}$; $\mathscr{F}(g)$ for $g=u, p, n$, $d^3=1$; $u^3=p^2=n^2=1$, $(up)^3=(pn)^2=1$, $(un)^2=d^{-1}$.
e_j, d, u, p, l	$\mathcal{R}_7$	$\mathscr{E}$; $\mathscr{F}(g)$ for $g=u, p, l$, $d^3=1$; $u^3=p^2=l^2=(up)^3=1$, $[p, l]=1$, $[u, l]=d$.
e_j, d, u, p, m	$\mathcal{R}_8$	$\mathscr{E}$; $\mathscr{F}(g)$ for $g=u, p, m$, $d^3=1$; $u^3=p^2=m^2=1$, $[d, u]=[d, p]=1$, $d^m=d^{-1}$, $(up)^3=(um)^2=(pm)^2=1$.
e_j, d, u, p	$\mathcal{R}_9$	$\mathscr{E}$; $\mathscr{F}(g)$ for $g=u, p$, $d^3=1$; $u^3=p^2=(up)^3=1$, $[d, u]=[d, p]=1$.
e_j, d, a, b, f	$\mathcal{R}_{11}$	$\mathscr{E}$; $\mathscr{F}(g)$ for $g=a, b, f$, $d^3=1$; $a^3=b^3=f^4=[a, d]=[b, d]=[f, d]=1$, $[a, b]=d$, $a^f=b$, $b^f=a^{-1}$.
e_j, d, u, m	$\mathcal{R}_{15}$	$\mathscr{E}$; $\mathscr{F}(g)$ for $g=u, m$, $d^3=1$; $u^3=m^2=[d, u]=1$, $d^m=d^{-1}$, $u^m=u^{-1}$.
e_j, d, u, l	$\mathcal{R}_{16}$	$\mathscr{E}$; $\mathscr{F}(g)$ for $g=u, l$, $d^3=1$; $u^3=l^2=[d, u]=1$, $d^l=d^{-1}$, $[u, l]=d$.

2.6. *Stabilizer Y_α*

By 2.3, the group $Y_\alpha/\langle d\rangle\cap Y_\alpha$ is conjugate in X/K to exactly one of F_i's above. Set $Z_\alpha=C_{Y_\alpha}(d)$. Then if the corresponding group F_i is not contained in A_6, Z_α is a subgroup of Y_α of index 2 satisfying $Z_\alpha/\langle d\rangle\cap Z_\alpha\simeq F_i\cap A_6$ and $Y_\alpha=Z_\alpha\langle g\rangle$ for any element g of Y_α inducing an odd permutation on $\mathscr{P}$. Furthermore, g inverts d. If F_i is contained in A_6 (this occurs iff we have N, N, N in the 4–6th columns of the table in 2.4), we have $Y_\alpha=Z_\alpha$ and $Z_\alpha/\langle d\rangle\cap Z_\alpha\simeq F_i$.

Since X is a non-split extension of S_6 by $\langle d\rangle$, any Sylow 3-subgroup of X_α does not have a complement for $\langle d\rangle$, Thus if $|F_i|_3=3^2$, that is, $i=1, 2, 10, 11, 12$ or 13, the group Y_α should contain $\langle d\rangle$, and the extension $Z_\alpha/\langle d\rangle$ does not split. On the other hand, if $|F_i|_3\leqslant 3$, since any subgroup of order 3^2 of X_α is elementary abelian, we may conclude that either Y_α does not contain d or the extension $Z_\alpha/\langle d\rangle$ splits.

2.7. *Stabilizer Y_β*

By Lemma 2.2, the stabilizer X_β of the line β in X is a split extension by the kernel $B=X^\infty_{\alpha,\beta}\simeq A_5$ on $\Gamma_0(\beta)$ of a complement $\langle e_1, e_1^d, d, s_5\rangle$ inducing S_4 on $\Gamma_0(\beta)$. The subgroup $\langle e_1, e_1^d, d\rangle\simeq A_4$ commutes with the kernel B. Now the stabilizer Y_β of the line β acts transitively on the set $\Gamma_0(\beta)$ of four points. Thus $Y_\beta/K_\beta\simeq S_4$, A_4, D_8, E_4 or 4, where $K_\beta=Y\cap B$ is the kernel of the action of Y_β on $\Gamma_0(\beta)$. Since Y_β contains a subgroup $\langle e_1, e_1^d\rangle$ of N by our assumption, we have $Y_\beta/K_\beta\not\simeq 4$.

Furthermore, if $Y_\beta/K_\beta \simeq S_4$ or D_8, there is a 2-element g of Y_β inducing a transposition on the four points $\Gamma_0(\beta)$. By flag-transitivity of Y, we may assume that g fixes the point α. Since d induces a 3-cycle fixing α on $\Gamma_0(\beta)$, g does not centralize d. Then g inverts d, and therefore the group F_i corresponding to $Y_\alpha/\langle d\rangle \cap Y_\alpha$ contains a 2-element inducing an odd permutation on $\Gamma_1(\alpha)$ fixing β (that is, of type 2^1 or 4^1).

Now note that $K_\beta \times \langle d\rangle$ coincides with the subgroup of $Y_{\alpha,\beta}$ inducing even permutations on $\Gamma_1(\alpha)$. If $Y_\alpha/\langle d\rangle \cap Y_\alpha$ is conjugate to a subgroup F_i of S_6 in the table of 2.4, then K_β is isomorphic to the stabilizer in the group $F_1 \cap A_6$ of the letter corresponding to β. Thus, it is given by the last column of the table.

We will show that $C_{Y_\beta}(K_\beta)$ contains $\langle e_1, e_1^d, d\rangle \simeq A_4$, if $Y_\beta/K_\beta \simeq S_4$ or A_4. Since Y_β contains a subgroup inducing A_4 on $\Gamma_0(\beta)$ and $\langle e_1, e_1^d\rangle \subseteq Y_\beta$, we may assume that Y_β contains a 3-element xd for some $x \in B$. As $|B|_3 = 3$, $x^3 = 1$. Suppose $x \neq 1$. Since xd fixes α and β, the image of x in the group $Y_\alpha/(Y_\alpha \cap \langle d\rangle)$ is a permutation of type 3^1 on the set $\Gamma_1(\alpha)$ of six lines. By the table in 2.4, it happens only when $|Y_\alpha/(Y_\alpha \cap \langle d\rangle)| \geqslant 3^2$. Then we have $d \in Y_{\alpha,\beta}$, as we noted in 2.5. Thus Y_β always contains a subgroup $\langle e_1, e_1^d, d\rangle$.

3. Some FEQs with Point-residues $T_2^*(O_4)$

3.1. *Notation.*

In this section and the next section, we assume that $\mathscr{G} = (\mathscr{G}_0, \mathscr{G}_1, \mathscr{G}_2; *)$ is an FEQ with point-residues $T_2^*(O_4)$ (see 2.2,3) or its dual $T_2^{**}(O_4)$, admitting a flag-transitive group G. Since $\mathscr{G}$ satisfies the (LL) property by our assumption, we may identify a line with two points incident with it. Let (P, l, u) be a maximal flag of $\mathscr{G}$ with $P \in \mathscr{G}_0$, $l = \{P, Q\} \in \mathscr{G}_1$ and $u \in \mathscr{G}_2$.

The group G_P/K_P is a flag-transitive subgroup of $X = \mathrm{Aut}(T_2^*(O_4))$. In the remainder of this paper, unless otherwise stated, we also assume that G_P/K_P contains a normal subgroup $O_2(X) = N \simeq 2^6$, acting regularly on the set of points of $T_2^*(O_4)$. However, note that there are several other flag-transitive subgroups of X and so some other possibilities for G_P, in general.

3.2. *Some elementary observations*

For any point R collinear with P, the image of K_R in G_P/K_P fixes all points (resp. lines) of $T_2^*(O_4)$, if $\mathscr{G}_P \simeq T_2^*(O_4)$ (resp. if $T_2^{**}(O_4)$) (see [10, Lemma 7(1)]). By 2.2, we have $K_P = K_R$, and therefore $K_P = 1$ by the connectivity of the point–line graph of $\mathscr{G}$. Thus the stabilizer G_P of a point P is isomorphic to a subgroup of X, acting flag-transitively on $T_2^*(O_4)$.

Since G_u acts transitively on the flags of type $\{0, 1\}$ on u, the group G_u/K_u is a doubly transitive permutation group on $\mathscr{G}_0(u)$.

3.3. *Stabilizer G_u*

In the remainder of this section, we assume that $\mathscr{G}_P \simeq T_2^*(O_4)$, and will prove Theorem 1.1. The sets $\mathscr{G}_1(P)$ and $\mathscr{G}_2(P)$ correspond to the sets of points and lines of $T_2^*(O_4)$, respectively, and therefore we may apply 2.6 and 2.7 to determine $G_{P,l}$ and $G_{P,u}$, respectively.

Since $\mathscr{G}_0(u) - \{P\}$ corresponds to the set of four points on the line u of $T_2^*(O_4)$, G_u/K_u acts doubly transitively on the set $\mathscr{G}_0(u)$ of five points by 3.2. Thus $(G_u/K_u, G_{P,u}/K) \simeq (S_5, S_4)$, (A_5, A_4) or $(F_5^4, 4)$. By 2.7, the last case does not occur and $\langle e_1, d\rangle \subseteq C_{G_{P,u}}(K_u)$. Furthermore, $K_l = \langle d\rangle$ by 2.6. Note that K_u coincides with the centralizer in $G_{P,Q} = G_{P,l}$ of $\langle e_1, e_1^d\rangle$.

Let M_u be the inverse image in G_u of the subgroup A_5 of G_u/K_u. Then the subgroup $\langle e_1, d\rangle$ inducing A_4 on $\mathscr{G}_0(u)$ is contained in M_u. Since $K_u \cap \langle e_1, d\rangle = 1$, the image of $C_{M_u}(K_u)$ in M_u/K_u is a normal subgroup of $M_u/K_u \simeq A_5$ containing a subgroup isomorphic to A_4. Then $M_u = K_u C_{M_u}(K_u)$.

Let $Q_1 := P$, $Q_2 := Q$, Q_i $(i = 3, 4, 5)$ be five points on u. By suitably numbering Q_i $(i = 3, 4, 5)$, we may assume that $d \in G_{P,l} = G_{P,Q}$ and $e_1 \in G_P - G_{P,l}$ induce permutations (Q_3, Q_4, Q_5) and $(Q, Q_3)(Q_4, Q_5)$, respectively. Since $M_u/K_u \simeq A_5$, there is an element of M_u inducing the permutation $(P, Q)(Q_4, Q_5)$. Let v be any such element. The element v satisfies the following relations modulo K_u:

$$v^2 \equiv 1, \qquad d^v \equiv d^{-1} \qquad \text{and} \qquad (e_1 v)^3 \equiv 1 \pmod{K_u}.$$

We may take v as a 2-element. Since $M_u/K_u \simeq A_5$ and $\langle e_1, e_1^d\rangle \cap K_u = 1$, a Sylow 2-subgroup S of M_u is a direct product of $\langle e_1, e_1^d\rangle$ and a Sylow 2-subgroup of K_u. Since K_u is a subgroup of $B \simeq A_5$, S is an elementary abelian group. Thus we have $v^2 = 1$. Since v stabilizes the line $l = \{P, Q\}$, v acts on $K_l = \langle d\rangle$. Thus $d^v = d^{-1}$.

By the last column of the table in 2.4, $K_u \simeq A_5$, F_5^2, S_3, 3, E_4, 2 or 1. If $K_u = 1$, v clearly satisfies the following set $\mathscr{B}$ of relations:

$$\mathscr{B}\colon\ v^2 = 1, \qquad d^v = d^{-1}, \qquad (e_1 v)^3 = 1.$$

Assume that $K_u \simeq 2$ or 2^2. Since $M_u = K_u C_{M_u}(K_u)$, as we have shown above, v centralizes K_u. Thus, by replacing v by xv for some $x \in K_u$ if necessary, we may assume that v satisfies the above set $\mathscr{B}$ of relations. Assume that $K_u \simeq A_5$, F_5^2 or S_3. Since $Z(K_u) = 1$ in these cases, we have $M_u = K_u \times C_{M_u}(K_u)$ and $C_{M_u}(K_u) \simeq A_5$. Thus $C_{M_u}(K_u)$ contains an element v inducing $(P, Q)(Q_4, Q_5)$. Since $Z(K_u) = 1$, v satisfies the relations $\mathscr{B}$. Consider the remaining case $K_u \simeq 3$. If $(e_1 v)^3 \neq 1$, the element $e_1 v$ is of order 9 fixing the points Q_4 and Q_5, which contradicts the fact that any 3-element of G_P is of order 3. Thus $(e_1 v)^3 = 1$ and v also satisfies the relations $\mathscr{B}$ in this case. Hence, in all cases, the relations $\mathscr{B}$ are satisfied.

3.4. *Stabilizer G_l*

We consider the permutation group G_l/K_l on the set $\mathscr{G}_2(l)$ of six planes. Note that $K_l = \langle d\rangle$ by 3.3. As in 2.3, we identify the letter i with the line through $[\mathbf{e}_1]$ and Q_i on the oval O_4 $(i = 1, \ldots, 6)$ in the residue $\mathscr{G}_P$, and in turn identify the corresponding plane of $\mathscr{G}_2(l)$. In particular, u corresponds to the letter 1. By suitably renumbering the remaining five planes of $\mathscr{G}_2(l)$, we may assume that $G_{P,Q}/K_l$ corresponds to the group F_i in the table of 2.4 for some $i = 1, \ldots, 16$.

In the remainder of this section, we say that G is of *type* F_i if $G_{P,Q}/K_l$ is a subgroup F_i of S_6 for some $i = 1, \ldots, 16$ in the table of 2.4. Since $K_l = \langle d\rangle$, it follows from 2.6 that the isomorphism type of G_P is uniquely determined for each type F_i.

The involution v of G_u (see 3.3) exchanges the two points P and Q on l, while fixing a plane u. Thus $G_l = G_{P,Q}\langle v\rangle$ and the image $\bar{v}$ of v in G_l/K_l is a permutation of type 1, 2^1 or 2^2, fixing the letter 1. Since K_u is centralized by v, $\bar{v}$ centralizes the image of K_u in $G_{P,Q}/K_l$, which is the stabilizer of 1 in F_i.

Assume that $\bar{v}$ is an identity permutation. Then $G_l/K_l = C_{G_l/K_l}(v) = C_{G_l}(v)K_l/K_l \simeq C_{G_l}(v)$, since the order of v is prime to $|K_l| = 3$ and $d^v = d^{-1}$. In particular, the extension G_l/K_l splits, and therefore $|G_l/K_l|_3 \leqslant 3$.

3.5. *Strategy for determining G and $\mathscr{G}$*

Since $G = \langle G_P, G_l, G_u\rangle = \langle G_P, v\rangle$ by the residually connectedness of $\mathscr{G}$, the most simple strategy to determine G and so $\mathscr{G}$ is as follows [10]. For each possible structure of G_P, we list up all the possible sets of relations in terms of v and generators of G_P, and then ask the computer to carry out the coset enumeration to determine the amalgamated product of G_P, G_u and G_l with respect to each possible set of relations. However, there are 16 possibilities for the structure of G_P, as noted in 3.4, and it may be too complicated if we simply follow the above method.

In order to make our procedure simpler, note that the following holds. Assume that a pair $(\mathscr{G}, G^i)$ of a simply connected FEQ $\mathscr{G}$ and a group G^i satisfy the hypothesis in this section for $(\mathscr{G}, G)$ for each $i = 1, 2$, and that G^i is of type F_{j_i} $(i = 1, 2)$ satisfying $F_{i_1} \subseteq F_{i_2}$. Then G_P^1 can be identified with a subgroup of G_P^2.

Thus, if we choose generators $g_1, \dots, g_n$ of G_P^2 so that $g_1, \dots, g_m$ $(m \leqslant n)$ generate G_P^1, each set $\mathscr{R}$ of possible relations among $g_1, \dots, g_n$ and v contains a set $\mathscr{S}$ of relations among $g_1, \dots, g_m$ and v. Then, if the amalgamated product of G_P^2, G_l^2 and G_u^2 with respect to the relations $\mathscr{R}$ acts flag-transitively on an FEQ $\mathscr{G}^2$, the subamalgamated product of G_P^1, G_l^1 and G_u^1 with respect to the relations $\mathscr{S}$ acts also flag-transitively on the same geometry $\mathscr{G}^2$, and therefore $\mathscr{G}^1 = \mathscr{G}^2$.

In particular, if we already know that the amalgamated product of G_P^1, G_l^1 and G_u^1 with respect to the relations $\mathscr{S}$ is too small to act flag-transitively on any FEQ in question, the relations $\mathscr{R}$ do not give any flag-transitive group on $\mathscr{G}^2$.

3.6. *Minimal cases*

By 3.5, it is natural to consider first the case in which $G_{P,Q}/K_l$ is a minimal transitive subgroup of S_6: that is, G is of type F_i for $i = 9$, 11, 15, 16 (see the table of 2.4). In this subsection, we will examine the groups G of these types.

Assume that G is of type F_{11}. Then G_P is generated by $e_1, e_2, e_3, d, a, b, f$ satisfying the presentation $\mathscr{R}_{11}$ by 2.5. Since v centralizes $K_u = C_{G_{P,Q}}(e_1, e_1^d)$ by 3.3, we have $b^v = b$. The permutation $\bar{v}$ normalizes both $G_{P,Q}/\langle d \rangle$ and $O_3(G_{P,Q})/\langle d \rangle = \langle (123), (456) \rangle$, it centralizes $\bar{b} = (456)$ and fixes the letter 1. Since $\bar{v}^2 = 1$, $\bar{v} = 1$ or (23), and so $f^v \equiv f$ or f^{-1} (mod $\langle d \rangle$). Since f is the unique element of order 4 in the coset $f\langle d \rangle$, we have $f^v = f$ if $\bar{v} = 1$ and $f^v = f^{-1}$ if $\bar{v} = (23)$. Furthermore, we have $a^v = ad^j$ if $\bar{v} = 1$ and $a^v = a^{-1}d^j$ if $\bar{v} = (23)$ for some $j = 0, \pm 1$. Thus the group G is generated by $e_1, e_2, e_3, d, a, b, f, v$ which satisfy one of the following relations $\mathscr{R}_{11}(i, j)$ for some $i = \pm 1$ and $j = 0, \pm 1$:

$$\mathscr{R}_{11}(i, j)\colon \mathscr{R}_{11}; \qquad \mathscr{B} \text{ in 3.3}; \qquad [b, v] = 1, \quad f^v = f^i \quad \text{and} \quad a^v = a^i d^j.$$

By coset enumeration using CAYLEY (version for the Sun 3), we may verify that the group with presentation $\mathscr{R}_{11}(i, j)$ collapses for any (i, j). Thus there is no group G of type F_{11}.

Next we consider the group G of type F_{16} (resp. F_{15}), in which G_P has a presentation $\mathscr{R}_{16}$ (resp. $\mathscr{R}_{15}$). The permutation $\bar{v}$ of order at most 2 fixing 1 acts on $O_3(G_{P,Q})/K_l = \langle \bar{u} \rangle = \langle (123)(456) \rangle$. Thus $\bar{v} = 1$ or (23)(56). Assume that the latter case holds. Then we have $u^v = u^{-1}d^i$ and $l^v = ld^j$ (resp. $m^v = md^j$) for some $i, j = 0, \pm 1$, since $\bar{l} = (14)(25)(36)$ (resp. $\bar{m} = (14)(26)(35)$). As $v^2 = 1$ and $d^v = d^{-1}$, we have $u = u^{v^2} = (ud^i)^{-1}d^{-i} = ud^i$ and so $i = 0$. Then we may verify that G of type F_{16} (resp. F_{15}) is generated by $e_1, e_2, e_3, d, u, l, v$ (resp. $e_1, e_2, e_3, d, u, m, v$) which satisfy one of the following relations $\mathscr{R}'_{16}(i)$ (resp. $\mathscr{R}'_{15}(i)$) for some $i = 0, \pm 1$:

$$\mathscr{R}'_{16}(i)\colon \mathscr{R}_{16}; \qquad \mathscr{B}; \qquad u^v = u^{-1}, \quad l^v = ld^i.$$

$$\mathscr{R}'_{15}(i)\colon \mathscr{R}_{15}; \qquad \mathscr{B}; \qquad u^v = u^{-1}, \quad m^v = md^i.$$

By coset enumeration, the group G^i (resp. H^i) with presentation $\mathscr{R}'_{16}(i)$ (resp. $\mathscr{R}'_{15}(i)$) satisfies $|G^i : G_P| = 25$, 1, 5 (resp. $|H^i : G_P| = 1$, 5, 25) respectively for $i = 0$, 1, -1. Since these values are smaller than 64, the number of points collinear with a point, we should have $\bar{v}=1$.

Then $u^v=ud^i$ and $l^v=ld^j$ (resp. $m^v=md^j$) for some $i,j=0, \pm 1$. Thus G of type F_{16} (resp. F_{15}) is generated by $e_1, e_2, e_3, d, u, l, v$ (resp. $e_1, e_2, e_3, d, u, m, v$) which satisfy one of the following relations $\mathcal{R}_{16}(i,j)$ (resp. $\mathcal{R}_{15}(i,j)$) for some $i,j=0, \pm 1$:

$$\mathcal{R}_{16}(i,j)\colon\ \mathcal{R}_{16};\qquad \mathcal{B};\qquad u^v=ud^i,\quad l^v=ld^j.$$

$$\mathcal{R}_{15}(i,j)\colon\ \mathcal{R}_{15};\qquad \mathcal{B};\qquad u^v=ud^i,\quad m^v=md^j.$$

By coset enumeration, the group $G^{(i,j)}$ with presentation $\mathcal{R}_{16}(i,j)$ collapses except for $(i,j)=(1,1)$, and $|G^{(1,1)}:G_P|=160$. The group $H^{(i,j)}$ with presentation $\mathcal{R}_{15}(i,j)$ contains the subgroup G_P of index 160 for $(i,j)=(0,1)$, $(-1,1)$ or $(1,-1)$, and collapses otherwise.

Finally, consider the group G of type F_9, in which G_P has a presentation $\mathcal{R}_9$ by 2.5. The permutation $\bar{v}$ of order at most 2 fixing the letter 1 acts on $O_2(G_{P,Q})/K_l=\langle\bar{p},\bar{p}^u\rangle=\langle(25)(36),(36)(14)\rangle$. Thus $\bar{v}=1$, $(25)(36)$, $(26)(35)$, $(23)(56)$ or $(25)(36)$. Since p lies in $K_u=C_{G_{P,Q}}(e_1,e_1^d)$, we have $[p,v]=1$ by 3.4. Since $\bar{u}=(123)(456)$, u^v coincides with u, u^{pu}, $u^{pu^{-1}}$, $(u^p)^{-1}$, u^{-1} or u^p modulo $\langle d\rangle$. Then we may verify that G is generated by $e_1, e_2, e_3, d, u, p, v$ which satisfy the set $\mathcal{R}(h,i)$ consisting of $\mathcal{R}_9$, $[p,v]=1$ and $u^v=u^hd^i$ for some $i=0,\pm 1$ and $h=\pm 1$, $\pm p$, pu or pu^{-1}. By coset enumeration, the group $G^{(h,i)}$ with presentation $\mathcal{R}(h,i)$ collapses or contains the subgroup G_P of index 20 or 25, except for $h=1$ and $i=0,1$. For $(h,i)=(1,0)$ and $(1,-1)$, we have $|G^{(h,i)}:G_P|=160$.

3.7. *Conclusion*

Suppose that G is of type F_i for $i=10$, 12 or 13. Since there is no group of type F_{11}, $i\neq 10$ by 3.5. Since F_i contains F_{16} for $i=12$, 13, it follows from 3.5 and 3.6 that v induces the trivial permutation on $\mathcal{G}_2(l)$. Since $|F_i|_3\geqslant 3$, it contradicts 3.4. Thus there is no group G of type F_i for $i=10$, 11, 12, 13.

Suppose that G is of type F_6. Since F_6 contains F_9, v acts trivially on $G_{P,Q}/K_l$. As $p,n\in K_u=C_{G_{P,Q}}(e_1,e_1^d)$, $[v,p]=[v,n]=1$. Then, by similar consideration to 3.6, G is generated by $e_1, e_2, e_3, d, u, p, n, v$ satisfying the relations $\mathcal{R}_6$, $\mathcal{B}$, $u^v=ud^i$, $[p,v]=[n,v]=1$ for some $i=0,\pm 1$. By coset enumeration, G collapses. Thus there is no group of type F_6. Similarly, we can verify that there is no group of type F_7. Since F_6 is contained in F_i for $i=1$, 2 or 5, there is no group G of type F_i ($i=1,2,5$) by 3.5.

Similarly, we may verify that the group G of type F_3 is generated by $e_1, e_2, e_3, d, u, p, q, m, v$ satisfying the set $\mathcal{R}_3(i,j)$ of relations $\mathcal{R}_3$, $\mathcal{B}$, $u^v=ud^i$, $p^v=p$, $q^v=q$, $m^v=md^j$ for some $i,j=0,\pm 1$, since $p\in K_u$ and q is the unique involution in the coset $q\langle d\rangle$. By coset enumeration, G collapses except for $(i,j)=(0,1)$ and the group $G^{(0,1)}$ with the presentation $\mathcal{R}_3(0,1)$ contains G_P of index 160.

Note that F_3 contains all the surviving cases F_i ($i=4$, 8, 9, 14, 15, 16). In the group G of type F_3, the element v satisfies the relations $u^v=u$ and $m^v=md$. Thus the subgroup H of G generated by $e_1, e_2, e_3, d, u, m, v$ is of type F_{15} satisfying the relations $\mathcal{R}_{15}(0,1)$. Now note that there are three conjugates of F_3 containing F_{15}; that is, $F_3^{b^k}$, for $k=0,1,2$. Thus each of three sets of relations $\mathcal{R}_{15}(0,1)$, $\mathcal{R}_{15}(-1,1)$ and $\mathcal{R}_{15}(1,-1)$ should be satisfied by F_{15} as a subgroup of $F_3^{b^k}$ for some $k=0,1,-1$. In fact, we may verify that $e_1, e_2, e_3, d, u, m, v^{b^k}$ satisfy the sets $\mathcal{R}_{15}(1,0)$, $\mathcal{R}_{15}(-1,1)$ and $\mathcal{R}_{15}(1,-1)$ of relations, respectively, for $k=0$, 1 and -1.

Since b fixes the letter 1, we obtain a group of type $F_3^{b^k}$ by renaming the letters $\{2,\ldots,6\}$; that is, $\mathcal{G}_2(l)-\{u\}$. Thus these three groups of type $F_3^{b^k}$ ($k=0,1,-1$)

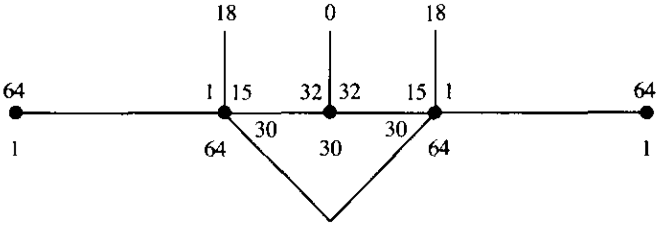

FIGURE 2. Orbit diagram for the point–line graph of $\mathscr{G}$.

determine isomorphic geometries, and therefore three presentations for groups of type F_{15} determine the isomorphic geometries.

Similarly, two presentations for the group of type F_9 give isomorphic geometries, because they determine groups of type F_3 and F_3^n, while F_3 and F_3^n contain F_9 and n fixes the letter 1.

3.8. Aut($\mathscr{G}$) *and some properties of* $\mathscr{G}$

The above presentation of $G = \mathrm{Aut}(\mathscr{G})$ of type F_3 gives a construction of $\mathscr{G}$ as a coset geometry of G by G_x ($x = P, l, u$). At present, the author does not know of any explicit presentation of our geometry $\mathscr{G}$ without referring groups. However, using CAYLEY, we can examine the permutation representation of G on the set of 160 points of $\mathscr{G}$.

Take the normal closure M in G of e_1e_2. Then, using the above permutation representation, we may verify that M is an extraspecial group of order 2^9 of plus type generated by $f_1 := e_1e_2$, $f_2 := f_1^d$, $f_3 := f_1^v$, $f_4 := f_1^{dv}$, $f_5 := f_1^{d^{-1}u}$, $f_6 := f_1^{du}$, $f_7 := f_1^{d^{-1}vu}$ and $f_8 := f_1^{dvu}$ satisfying $[f_i, f_j] = 1$ except for $i + j = 9$ and $[f_i, f_{9-i}]$ is the unique central involution z for $i = 1, \ldots, 8$. By the relations satisfied by G, we have $G/M \simeq (A_5 \times A_5) \,.\, 2$, where the images of $\langle e_1, d, v\rangle$ and $\langle u, p, q\rangle$ correspond to mutually commuting A_5-subgroups. By examining the centralizer of elements of order 5, we can verify that the extension G/M does not split. The subgroup $\langle e_1, d, v\rangle$ (resp. $\langle u, p, q\rangle$) stabilizes two mutually disjoint maximal totally isotropic subspaces $\langle f_1, f_2, f_3, f_4, z\rangle/\langle z\rangle$ and $\langle f_5, f_6, f_7, f_8, z\rangle/\langle z\rangle$ (resp. $\langle f_1, f_2, f_5, f_6, z\rangle/\langle z\rangle$ and $\langle f_3, f_4, f_7, f_8, z\rangle/\langle z\rangle$) of the 8-dimentional orthogonal space $M/\langle z\rangle$ of plus type.

Using CAYLEY, we may verify that there are five orbits of lengths 1, 1, 30, 64 and 64 on points under the action of the stabilizer G_P of a point P and that there are seven (resp. three) G_P-orbits on lines (resp. planes) of lengths $2, 2, 18, 18, 30, 30$ and 60 (resp. 5, 5 and 150). Furthermore, we have the orbit diagram of the point–line graph of $\mathscr{G}$ given in Figure 2. The quotient group $G/\langle z\rangle$ also acts flag-transitively on the quotient geometry $\bar{\mathscr{G}}$ of 80 points, with the orbit diagram of the point–line graph shown in Figure 3. In particular, $\mathscr{G}$ satisfies the (BH) property ([4, p. 255]); that is, any mutually collinear three points are incident with a plane, but $\bar{\mathscr{G}}$ does not.

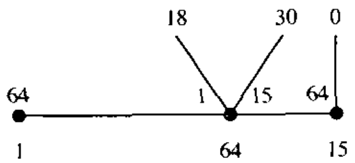

FIGURE 3. Orbit diagram for the point–line graph of $\bar{\mathscr{G}}$.

4. Some FEQs with Point-Residues $T_2^{**}(O_4)$

We follow the notation in 3.1,2. In this subsection, we assume that $\mathscr{G}_P \simeq T_2^{**}(O_4)$, the dual of $T_2^*(O_4)$, and will prove Theorem 1.2. Arguments are similar to those in 3.4–3.7.

4.1. *Stabilizers G_u and G_l*

Since $\mathscr{G}_P \simeq T_2^{**}(O_4)$, $\mathscr{G}_1(P)$ and $\mathscr{G}_2(P)$ correspond to the sets of lines and points of $T_2^*(O_4)$ respectively, and therefore we may apply 2.6 and 2.7 to determine $G_{P,u}$ and $G_{P,l}$ respectively. We may identify the plane $u \in \mathscr{G}_2(P)$ and the line $l \in \mathscr{G}_1(P, u)$ with the point α and the line β in $T_2^*(O_4)$ (see 2.3). Then each line of $\mathscr{G}_1(P, u)$ can be identified with a line of $T_2^*(O_4)$ joining α and Q_i for some $i = 1, \ldots, 6$, where Q_i are six points in the oval O_4 (see 2.2). We also denote by Q_i the unique point different from P on the line of $\mathscr{G}_1(P, u)$ corresponding to the line joining α and Q_i $(i = 1, \ldots, 6)$. When we consider the action of G_u on the set $\mathscr{G}_0(u)$ of seven points (or the set $\mathscr{G}_1(u)$ of pairs of these points), we simply denote Q_i by i $(i = 1, \ldots, 6)$ and denote P by 0. In particular, the point Q and the line $l = \{P, Q\}$ correspond to 1 and $\{0, 1\}$, respectively, since β is the line joining α and Q_1.

Since $\mathscr{G}_0(u) \times \{P\}$ corresponds to the six lines on the point u of $T_2^*(O_4)$, G_u/K_u acts doubly transitively on the set $\mathscr{G}_0(u)$ of seven points by 3.2. Thus $(G_u/K_u, G_{P,u}/K_u) \simeq (S_7, S_6)$, (A_7, A_6), $(L_3(2), S_4)$ or $(F_7^6, 6)$. Then we may assume that $G_{P,u}/K_u$ corresponds to F_1, F_2, F_6 and F_{16} of S_6, respectively, where F_i are transitive subgroups of S_6 in the table in 2.4. (Note that if $G_u/K_u \simeq L_3(2)$, $G_{P,u}/K_u \simeq S_4$ corresponds to the subgroup F_6, since $L^3(2)$ and sc its one point stabilizer S_4 consists of even permutations on seven points.)

As in Section 3, we say that G is of *type* F_i if $G_{P,u}/K_u$ is F_i $(i = 1, 2, 6, 16)$. By 2.6, we have $K_u = 1$ or $K_u = \langle d \rangle \simeq 3$. If G is of type F_1 or F_2, we have $K_u \simeq 3$, since $G_{P,u}$ is a subgroup of $X = \mathrm{Aut}(T_2^*(O_4))$, a non-split triple extension of S_6, and $|G_{P,u}/K_u|_3 = 3^2$. Then $G_u \simeq 3S_7$ or $3A_7$. If G is of type F_{16}, it follows from 2.6 that either $G_u \simeq 6$ and $K_u = 1$ or $G \simeq (3 \times 3)2$ and $K_u = \langle d \rangle$ is inverted by involutions of $G_{P,u}$. If G is of type F_6, G_u is a direct product of $L_3(2)$ and K_u, since the Schur multiplier of $L_3(2)$ is of order prime to 3.

Since l corresponds to the line β of $T_2^*(O_4)$ through the point u, the group $G_{P,Q} = G_{P,l}$ is a subgroup of X_β (see 2.7) containing $\langle e_1, e_1^d \rangle$. By 2.7, we have $K_l \simeq A_5$ if G is of type F_1 or F_2, $K_l \simeq E_4$ if G is of type F_6, and $K_l = 1$ if G is of type F_{16}.

If G is of type F_1 or F_2, $M := C_{G_{P,l}}(K_l)$ contains $\langle e_1, e_1^d, d \rangle$ and $M \cap K_l = 1$. Then $G_{P,l}/K_l \simeq A_4$ or S_4, and so we have $\langle e_1, e_1^d \rangle = O_2(M)$, a characteristic subgroup of $G_{P,l}$. Assume that G is of type F_{16}. Since $G_{P,u} \simeq 6$ does not contain an element inducing a permutation on $\mathscr{G}_0(u) - \{P\}$ of type 2^1 or 4^1 by the table in 2.4, it follows from 2.7 that $G_{P,l} \simeq E_4$ or A_4. Then we have $\langle e_1, e_1^d \rangle = O_2(G_{P,l})$, a characteristic subgroup of $G_{P,l}$. Since G_l contains $G_{P,Q} = G_{P,l}$ as a subgroup of index 2, any element v of G_l acts on $G_{P,l}$ and so on $\langle e_1, e_1^d \rangle$.

Assume that G is of type F_6. Since $\mathscr{G}_0(u)$ and $\mathscr{G}_1(u)$ correspond to the projective points and the pairs of points of a projective plane over $\mathbf{F}_2$, the stabilizer of l in $G'_u \simeq L_3(2)$ is isomorphic to D_8. In particular, there is an involution v in $G'_{u,l} - G'_{u,P,Q}$. Since K_u is a four group in G'_u, $G_{u,l}$ induces the identity permutation or a transposition on the three planes of $\mathscr{G}_2(l) - \{u\}$. (In 4.2, we will show that the former case holds). Since $G_{P,u}/K_u \simeq F_6$ consists of even permutations, then it follows from 2.7 that $G_{P,l}/K_l \simeq E_4$ or A_4. Thus $G_{P,Q} = K_l \times \langle e_1, e_1^d \rangle$ or $K_l \times \langle e_1, e_1^d, d \rangle$ by the last claim in 2.7. Since v acts on $O_2(G_{P,l}/K_l)$, we have $[e_1^{d^i}, v] = x$ for some $x \in K_l$ and some $i = 0, \pm 1$. As $v^2 = 1$, we have $[x, v] = 1$.

Furthermore, assume that $K_u = \langle d \rangle$. Then, as $[d, v] = 1$ and $[d, K_l] = 1$, we have $e_1^{d^i} = e_1^{d^i}x$ for any $i = 0, 1, -1$. Since $e_1 e_1^d e_1^{d^{-1}} = 1$, we have $x = 1$. Thus v centralizes $\langle e_1, e_1^d \rangle$, and induces the identity permutation on $\mathscr{G}_2(l)$.

4.2. *The determination of G and $\mathscr{G}$*

We consider each case separately. We first consider the group of type F_{16} with $d \notin G_{P,Q}$. Then G_P is generated by elements $e_k, f_k := e_k^d$ $(k = 1, 2, 3)$, u and l which satisfy a presentation $\mathscr{R}_{16}$ in 2.5, ignoring the relations containing d and the relations $e_k e_k^d e_k^{d^{-1}} = 1$ $(k = 1, 2, 3)$. By 4.1, we have $K_l = K_u = 1$, $G_u \simeq F_7^6$ and $G_{P,Q} = \langle e_1, e_1^d \rangle \simeq E_4$. Since $G_u \simeq F_7^6$ and $G_{P,u} \simeq 6$, there is an involution v of G_u such that vl is of order 7 and $(vl)^{ul} = (vl)^3$. Conversely, the relations $\mathscr{B}_{16}$: $v^2 = u^3 = l^2 = (vl)^7 = [u, l] = 1$ and $(vl)^{ul} = (vl)^3$ give a presentation for F_7^6.

By taking a conjugate by ul, we may assume that v induces the cycle $(0, 1)$ on $\mathscr{G}_0(u)$: that is, v exchanges two points P and Q on l, and so acts on $G_{P,Q}$. Thus the relations $\mathscr{T}_{16}(i)$: $(e_1^{d^i})^v = e_1^{d^i}$, $(e_1^{d^{i+1}})^v = e_1^{d^{i-1}}$ hold for some $i = 0, 1, -1$ or the relations $\mathscr{T}_{16}(\infty)$: $e_1^v = e_1$, $(e_1^d)^v = e_1^d$ are satisfied.

By coset enumeration, the group $G^{(i)}$ with generators e_j, e_j^d $(j = 1, 2, 3)$, u, l and v satisfying the relations $\mathscr{R}_{16} \cup \mathscr{B} \cup \mathscr{T}_{16}(i)$ contains the subgroup G_P of index 7, 896, 7 and 7, respectively for $i = 0, 1, -1$ and ∞. Since there are 96 points collinear with P, we should have $i = 1$ and $|\mathscr{G}_0| = |G^{(1)} : G_P| = 896$.

Next we consider the group G of type F_6 with $d \notin G_{P,Q}$. By 4.1, $K_u = 1$, $G_u \simeq L_3(2)$, $K_l = \langle p, n \rangle$ and $G_{P,Q} = K_l \times \langle e_1, e_1^d \rangle \simeq E_4 \times E_4$. Since $u_1 := d^{-1}u$, p and n satisfy the standard presentation $(v_1v_2)^3 = (v_2v_3)^3 = (v_1v_3)^2 = v_i^2 = 1$ $(i = 1, 2, 3)$ for S_4, G_P is generated by e_j, e_j^d $(j = 1, 2, 3)$, $u_1 := d^{-1}u$, p and n with presentation $\mathscr{R}_6$ in 2.5.

We may identify the six points of $\mathscr{G}_0(u)$ with some projective points of the projective plane over $\mathbf{F}_2$, in which the points $1, \ldots, 6$ of $\mathscr{G}_0(u)$ correspond respectively to $(0,1,0)$, $(0,0,1)$, $(0,1,1)$, $(1,1,0)$, $(1,0,1)$ and $(1,1,1)$, and the point P corresponds to $(1,0,0)$. Then the permutations $u_1 = (123)(456)$, $p = (25)(36)$ and $n = (23)(56)$ of $G_{P,u}$ correspond to the following matrices in $L_3(2)$—we also take the following involution v of $L_3(2)$:

$$u_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad v := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the involution v induces the permutation $(01)(35)$ on the seven points of $\mathscr{G}_0(u)$, where 0 corresponds to P. Thus v acts on the line $l = \{P, Q\} = \{0, 1\}$. We may verify that the following relations are satisfied by u_1, p, n and v and that they give a presentation for $L_3(2)$:

$$\mathscr{B}_6\colon\ u_1^3 = p^2 = n^2 = v^2 = (u_1p)^3 = (u_1n)^2 = (pn)^2 = 1, \qquad p^v = n \qquad \text{and} \qquad (vu_1p)^3 = 1.$$

In particular, $C_{K_u}(v) = \langle pn \rangle$. By 4.1, we have $(e_1^{d^i})^v = e_1^{d^i}x$ for some $i = 0, 1, -1$ and $x = 1$ or pn. By replacing v by d^ivd^{-i}, we may assume that $e_1^v = e_1x$, since $[\langle u_1, p, n \rangle, d] = 1$. Suppose that v acts non-trivially on $G_{P,Q}/K_l$. Then $f_1^v = e_1f_1y$ for some $y \in K_l$, where $f_1 := e_1^d$. As $v^2 = 1$, $y \in C_{K_l}(v) = \langle pn \rangle$. Thus, in this case, G is generated by $e_k, f_k := e_k^d$ $(k = 1, 2, 3)$, u_1, p, n and v satisfying the relations $\mathscr{R}_6 \cup \mathscr{B}_6 \cup \mathscr{T}_6'(i, j)$ for some $i, j = 0, 1$, where $\mathscr{T}_6'(i, j)$ is the following set of relations:

$$\mathscr{T}_6'(i, j)\colon\ [e_1, v] = (pn)^i \qquad \text{and} \qquad f_1^v = e_1f_1(pn)^j.$$

By coset enumeration, the group $G^{(i,j)}$ with presentation $\mathscr{R}_6 \cup \mathscr{B}_6 \cup \mathscr{T}_6(i; j)$ collapses for $i = 1$ and contains the subgroup G_P of index 14 for $i = 0$. Thus v centralizes

$G_{P,Q}/K_l$. Then the group G is generated by e_k, f_k $(k=1,2,3)$, u_1, p, n and v satisfying the relations $\mathcal{R}_6 \cup \mathcal{B}_6 \cup \mathcal{T}_6(i,j)$ for some $i, j = 0, 1$, where $\mathcal{T}_6(i,j)$ is the following set of relations:

$$\mathcal{T}_6(i,j)\colon\ [e_1, v] = (pn)^i \qquad \text{and} \qquad [f_1, v] = (pn)^j.$$

Since u_1, p, n commute with d, we may cyclically permute e_k, f_k and $e_k f_k$ for each $i = 1, 2, 3$. Thus we may assume that $i = 0$. By coset enumeration, the group G^j with presentation $\mathcal{R}_6 \cup \mathcal{B}_6 \cup \mathcal{T}_6(0,j)$ contains the subgroup G_P of index 896 for $j = 0$ and 448 for $j = 1$.

Similarly, if G is of type F_6 and $d \in G_{P,Q}$, it follows from the above arguments and the last remark in 4.1 that G has a presentation $\mathcal{R}_6 \cup \mathcal{B}_6 \cup \mathcal{T}_6(0,0)$, and therefore $|G : G_P| = 896$.

Now we consider the largest case in which G is of type F_1. By 4.1, $G_P = X \simeq 2^6 3S_6$ is generated by e_k $(k=1,2,3)$, d, s_i $(i=1,\ldots,6)$ satisfying the relations $\mathcal{R}_1$. Furthermore, $G_u \simeq 3S_7$ with $K_u = \langle d \rangle$. We choose an involution s_0 of G_u inducing the transposition (P, Q) on the seven points on u; that is, inducing (01). Then it satisfies the following relations:

$$s_0^2 = 1, \quad d^{s_0} = d^{-1}, \quad (s_0 s_1)^3 = 1, \quad (s_0 s_i)^2 \equiv 1 \qquad (\mathrm{mod}\langle d \rangle) \text{ for } i = 1, \ldots, 5.$$

By coset enumeration, we can verify that there are three possible relations presenting $3S_7$ which satisfy the above relations modulo $\langle d \rangle$. (Other possible relations generate S_7 only.) Taking conjugates by d, they are equivalent to the following relaitons:

$$\mathcal{S}_0\colon\ s_0^2 = 1, \qquad d^{s_0} = d^{-1}, \qquad (s_0 s_1)^3 = 1, \qquad (s_0 s_i)^2 = 1 \quad \text{for } i = 1, \ldots, 5.$$

By 4.1, s_0 acts on $O_2(G_{P,Q}) = \langle e_1, e_1^d \rangle$. Thus there are three possibilities, $\mathcal{T}_1(i)$ $(i = 0, 1, 2)$, between s_0 and e_1:

$$\mathcal{T}_1(i)\colon\ [e_1^{d^i}, s_0] = 1.$$

Then the group G satisfies the relations $\mathcal{R}_1 \cup \mathcal{S}_0 \cup \mathcal{T}_1(i)$ (see 2.5) for some $i = 0, 1, 2$. By coset enumeration using CAYLEY, the group $G^{(i)}$ with presentation the above relations can be determined. We have $|G^{(i)} : G_P| = 7$ for $i = 1, 2$ but $|G^{(0)} : G_P| = 896$.

Finally, consider the group G of type F_2. Then G_P is generated by elements e_i $(i = 1, 2, 3)$, and $t_j := s_j s_5$, inducing the transposition $(j, j+1)(5, 6)$ on $\mathcal{G}_0(u)$ $(j = 1, \ldots, 4)$. with presentation $\mathcal{R}' \cup \mathcal{F}'$, where $\mathcal{R}'$ and $\mathcal{F}'$ are as follows (note that $\mathcal{R}'$ give a presentation for the non-split central extension $3A_6$):

$$\begin{aligned}
\mathcal{R}'\colon\ & [d, t_i] = 1 \quad (i = 1, \ldots, 4), \qquad t_j^2 = t_4^3 = (t_j \cdot t_{j+1})^3 = 1 \quad (j = 1, 2, 3),\\
& (t_i \cdot t_j)^2 = 1 \quad \text{for any } 1 \leqslant i < j \leqslant 4 \text{ with } j - i \geqslant 2\\
& \text{except for } (i, j) = (1, 4), \qquad (t_1 \cdot t_4)^2 = d^{-1}.\\
\mathcal{F}'\colon\ & \mathcal{E} \text{ in 2.5 and } e_1^{t_1} = e_2, \qquad e_3^{t_1} = e_3, \qquad e_1^{t_2} = e_1, \qquad e_2^{t_2} = e_3,\\
& e_1^{t_3} = e_1, \qquad e_2^{t_3} = e_2, \qquad e_3^{t_3} = e_1 e_2 e_3,\\
& e_1^{t_4} = e_1, \qquad e_2^{t_4} = e_2^{d^{-1}}, \qquad e_3^{t_4} = e_3^d.
\end{aligned}$$

By a similar argument to the above, there is an involution t_0 of G_l acting on $O_2(G_{P,Q}) = \langle e_1, de_1^d \rangle$ satisfying the following relations $\mathcal{A}$ (t_0 corresponding to the permutation (01) (23) on $\mathcal{G}_0(u)$): $\mathcal{A}$: $t_0^2 = 1$, $[d, t_0] = 1$, $(t_0 t_1)^3 = 1$ and $(t_0 t_i)^2 = 1$, and $\mathcal{T}$: $[e_1, t_0] = 1$. By coset enumeration, the group G with presentation $\mathcal{R}' \cup \mathcal{F}' \cup \mathcal{A} \cup \mathcal{T}$ contains the subgroup G_P of index 896.

Now we can complete the proof of Theorem 1.2. As is observed in 3.4, for any $i = 1, 2, 6, 16$, the group $G^{(0)}$ of type F_1 with presentation $\mathcal{R}_1 \cup \mathcal{S}_0 \cup \mathcal{T}_1(i)$ above contains a subgroup of type F_i acting flag-transitively on the simply connected coset geometry $\mathcal{G}^{(0)}$ for $G^{(0)}$. By the above argument, we observed that in each case except when G is of type F_6 and $d \notin G_{P,Q}$, there is a unique set of presentations for G, and that if G is of type F_6 and $d \notin G_{P,Q}$, we have two possible sets of presentations for G. Thus, except for this case, any group in question is a subgroup of $G^{(0)}$ acting flag-transitively on the simply connected geometry $\mathcal{G}^{(0)}$ for $G^{(0)}$ with 896 points. In the exceptional case, we have the other simply connected geometry $\mathcal{G}^{(1)}$ defined on 448 points with the full automorphism group G for which $\mathcal{R}_6 \cup \mathcal{B}_6 \cup \mathcal{T}_6(0, 1)$ gives a presentation. We may construct two geometries $\mathcal{G}^{(i)}$ $(i = 0, 1)$ in terms of coset geometries, and observe that they are FEQs (see also 4.3,4). Thus theorem 1.2 is proved.

4.3. *Properties of* $\mathcal{G}^{(i)}$ $(i = 0, 1)$

The permutation representations of the groups $G^{(i)}$ $(i = 0, 1)$ on the set of points are obtained by coset enumeration. Thus we can work with them with the aid of CAYLEY, and verify the following.

The group $G := G^{(0)}$ is a split extension of $G_u = \langle d, s_i, v \mid i = 0, \ldots, 5\rangle \simeq 3S_7$ by the largest normal 2-subgroup M, of order 2^{13}. The group M is an extraspecial group of plus type. The unique central involution z is given by the product of $e_1^{s^i}$ for $i = 0, \ldots, 6$ in this order, where $s = s_0 s_1 \cdots s_5$ is an element of order 7. There is a quotient geometry $\mathcal{G}^{(0)}/\langle z\rangle$ on 448 points with the full automorphism group $G/\langle z\rangle \simeq 2^{12}:3S_7$. There are seven (resp. five) orbits of G_P (resp. G_u) on the set of 896 points of $\mathcal{G}^{(0)}$ of lengths 1, 1, 36, 90, 96, 96 and 576 (resp. 7, 7, 252, 315 and 315). We may verify the orbit diagrams of the point–line graphs of $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(0)}/\langle z\rangle$ given in Figures 4 and 5. In particular, both $\mathcal{G}$ and $\bar{\mathcal{G}}$ satisfy the (BH) property ([4, p. 255]); that is, any mutually collinear three points are incident with a plane.

The group $H := G^{(1)}$ is a split extension of $H_u = \langle u_1, p, n, v\rangle$ by the largest normal 2-subgroup M of order 2^{12}. The group M is a non-abelian special group with center of order 2^6, in which the normal closure of e_1 is an elementary abelian subgroup of M of order 2^9. (In particular, this implies that H is not a subgroup of the full automorphism group $G/\langle z\rangle \simeq 2^{12}: 3S_7$ on the quotient geometry $\mathcal{G}^{(0)}/\langle z\rangle$, and therefore the geometry $\mathcal{G}^{(1)}$ is not isomorphic to $\mathcal{G}^{(0)}/\langle z\rangle$.) There are 13 (resp. 11) orbits $\mathcal{O}(i)$ of H_P (resp. H_u) on the set of 448 points of $\mathcal{G}^{(0)}$ of lengths $|\mathcal{O}(i)| = 1, 3, 3, 3, 4, 6, 12, 16, 16, 96, 96, 96$ and 96, respectively for $i = 1, \ldots, 13$, in which the orbit $\mathcal{O}(12)$ consists of points collinear with P. There are 11 orbits of H_u on $\mathcal{G}^{(0)}$ of lengths 7, 21, 21, 21, 28, 28, 28, 42, 84, 84 and 84.

Instead of presenting the orbit diagram of the point–line graphs of $\mathcal{G}^{(1)}$, we give the

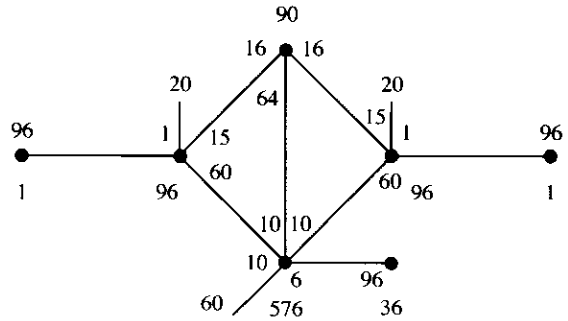

FIGURE 4. Orbit diagram for the point–line graph of $\mathcal{G}^{(0)}$.

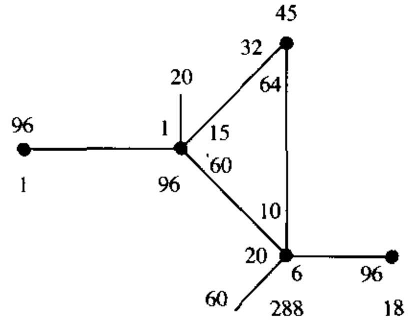

FIGURE 5. Orbit diagram for the point–line graph of $\mathscr{G}^{(0)}$.

numbers $a(i, j)$ of points in $\mathcal{O}(j)$ collinear with a fixed point in $\mathcal{O}(i)$ for $1 \leqslant i \leqslant j \leqslant 11$:

$a(1, 12)) = 96$, $a(2, 12) = 32$, $a(2, 13) = 64$, $a(3, 10) = 64$,
$a(3, 12) = 32$, $a(4, 11) = 64$, $a(4, 12) = 32$, $a(5, 11) = 48$,
$a(5, 13) = 48$, $a(6, 10) = 32$, $a(6, 11) = 32$, $a(6, 13) = 32$,
$a(7, 10) = 32$, $a(7, 11) = 16$, $a(7, 12) = 32$, $a(7, 13) = 16$,
$a(8, 10) = 24$, $a(8, 11) = 24$, $a(8, 12) = 24$, $a(8, 13) = 24$,
$a(9, 10) = 24$, $a(9, 11) = 24$, $a(9, 12) = 24$, $a(9, 13) = 24$,
$a(10, 10) = 20$, $a(10, 11) = 16$, $a(10, 12) = 20$, $a(10, 13) = 24$,
$a(11, 11) = 20$, $a(11, 12) = 24$, $a(11, 13) = 20$, $a(12, 12) = 20$,
$a(12, 13) = 16$ and $a(13, 13) = 20$.

Furthermore, we can verify that the stabilizer $G_{P,Q}$ has two orbits of lengths 16 and 4 on the set of points collinear with P and Q. Thus $\mathscr{G}^{(1)}$ does not satisfy the (BH) property.

4.4. *A construction of the FEQ* $\mathscr{G}^{(0)}$

In this subsection, we will give a brief construction of the FEQ $\mathscr{G}^{(0)}$ on 896 points for the group $G^{(0)}$, in terms of vector spaces. Let V be an 8-dimensional vector space over $\mathbf{F}_4$ with a unitary form h, and H a hyperplane perpendicular to an isotropic point, say P_0. We take for the set $\mathscr{G}_2$ of *planes* the set of isotropic points outside from H. The vectorwise stabilizer of P_0 in $SU_8(2)$ (isomorphic to $2_+^{1+12}{:}SU_6(2)$) has a normal group $K \simeq 2_+^{1+12}$ acting regularly on $\mathscr{G}_2$.

The subgroup U of $SU_8(2)$ fixing P_0 vectorwise and stabilizing a plane $P \in \mathscr{G}_2$ is isomorphic to $SU_6(2)$. By the explicit construction given below, we may verify that there are seven 4-dimensional isotropic subspaces X_i $(i = 1, \ldots, 7)$ containing P and a subgroup $A \simeq 3A_7$ satisfying the following properties:

(1) The group A acts doubly transitively on $\{X_i \mid i = 1, \ldots, 7\}$.
(2) $\dim(X_i \cap H) = 3$ and $\dim(X_i \cap X_j \cap H) = 1$ for any distinct $1 \leqslant i, j \leqslant 7$.
(3) For any fixed X_i, the six 1-dimensional subspaces $(X_i \cap X_j \cap H)(1 \leqslant j \neq i \leqslant 7)$ form an oval in the projective space $H \cap X_i$.

For example, we take a unitary form $h(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{8} x_i y_{9-i}^2$ on $V = \mathbf{F}_4^8$, $P_0 = [\mathbf{e}_1] = [(1, 0, 0, 0, 0, 0, 0, 0)]$ and $P = [\mathbf{e}_8] = [(0, 0, 0, 0, 0, 0, 0, 1)]$. Then the vectorwise stabilizer of P_0 in $SU_8(2)$ consists of the following matrices $M(\alpha, \mathbf{a}; X)$ for $\alpha \in \mathbf{F}_4$, $\mathbf{a} \in \mathbf{F}_4^6$ with $\alpha + \alpha^2 = h(\mathbf{a}, \mathbf{a})$ and $X \in SL_6(4)$ with $XJ_6\bar{X}^{\mathrm{t}} = J_6$, where J_6 is the anti-diagonal matrix with the (i, j)-entry 1 for $i + j = 7$ and 0 otherwise, and $\bar{X}$ and X^{t} denote the algebraic conjugate and the transpose of X, respectively:

$$M(\alpha, \mathbf{a}; X) = \begin{pmatrix} 1 & \mathbf{0} & 0 \\ XJ_6\bar{\mathbf{a}}^t & X & \mathbf{0}^t \\ \alpha & \mathbf{a} & 1 \end{pmatrix}.$$

The subgroup $K := \{M(\alpha, \mathbf{a}; I_6) | \alpha \in \mathbf{F}_4, \mathbf{a} \in \mathbf{F}_4^6, \alpha + \alpha^2 = h(\mathbf{a}, \mathbf{a})\}$ is the kernel on the isotropic points of $H = P_0^{\perp}$, isomorphic to 2_+^{1+12} and $U := \{M(0, \mathbf{0}; X) \mid X \in SL_6(4), XJ_6\bar{X}^t = J_6\}$ is a subgroup of $SU_8(2)$ fixing P_0 vectorwise and stabilizing P, isomorphic to $SU_6(2)$.

We consider the subgroup A of U generated by the following matrices, where ω is a generator of $\mathbf{F}_4^*$, $\bar{\omega} = \omega^2$ and we denote an element $M(0, \mathbf{0}; X)$ of U simply by X:

$$t_1 = \begin{pmatrix} 1 & 0 & 0 & \bar{\omega} & \bar{\omega} & 0 \\ 0 & 0 & 1 & 1 & 1 & \omega \\ 0 & 1 & 0 & 1 & 1 & \omega \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$t_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & \omega & 0 & 0 \\ 1 & 0 & 1 & 1 & \bar{\omega} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad t_4 = \begin{pmatrix} \omega & 0 & 0 & 1 & 1 & 0 \\ 0 & \bar{\omega} & 0 & \bar{\omega} & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & \omega \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{\omega} & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \end{pmatrix},$$

$$t_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & \omega & 0 & 0 \\ \bar{\omega} & \bar{\omega} & 1 & 0 & \bar{\omega} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & \omega & 1 & 0 \\ 0 & 1 & 0 & \omega & 1 & 1 \end{pmatrix}.$$

We may verify that t_i $(i = 0, \ldots, 4)$ satisfy the relations $\mathcal{R}'$ (see 4.2), giving a presentation for the non-split central extension $3A_7$, by taking d as the diagonal matrix ωI_6. Furthermore, $t_1, \ldots, t_4$ stabilizes the subspace $X_1 := \langle \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8 \rangle$ of V. Thus the set of the conjugates of X_1 under A consists of seven subspaces of dimension 4, on which A acts doubly transitively. We have $X_1^{t_0} \cap X_1 = \langle \mathbf{e}_5, \mathbf{e}_8 \rangle$. Since t_i induces on X_1 the same action as g_i in 2.2 $(i = 1, \ldots, 6)$, the orbits of $\langle t_i \mid i = 1, \ldots, 6 \rangle$ containing $[\mathbf{e}_5]$ form an oval O_4 on the 3-dimensional space $X_1 \cap H = \langle \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7 \rangle$. Thus all the properties (1)–(3) above are satisfied by A and X_i $(i = 1, \ldots, 7)$, which are the conjugates of X_1 by A.

Now we define $\mathcal{G}_0$ and $\mathcal{G}_1$ to be the set of seven spaces X_i of dimension 4 and $\binom{7}{2} = 21$ spaces $X_i \cap X_j$ of dimension 2 $(1 \leq i \neq j \leq 7)$ together with their conjugates by $K \simeq 2^{1+12}$, respectively. Incidence is determined by inclusion.

By the construction, the group $KA \simeq 2_+^{1+12}{:}3A_7$ acts on $\mathcal{G}$. Using the properties (1)–(3) above, we can verify that $\mathcal{G}$ is an FEQ, admitting a flag-transitive group KA as follows. First, we examine the residue at a plane u. Since K acts regularly on $\mathcal{G}_2$, we may take $u = P$. For the plane $P = [\mathbf{e}_8] \in \mathscr{G}_2$, the residue at P consists of 7 points X_i and 21 lines $X_i \cap X_j \in \mathscr{G}_1$, so that it forms a circle geometry with seven vertices. Furthermore, A acts flag-transitively on $\mathscr{G}_P$ by (1). Then, KA acts flag-transitively on $\mathscr{G}$, and so we may take the point X_1 to examine the residues at points. For the point $X_1 \in \mathscr{G}_0$, the planes of $\mathscr{G}_2$ incident with X_1 correspond to isotropic points of the projective space X_1 outside from the projective plane $X_1 \cap H$. Each line of $\mathscr{G}_1$ incident with X_1 corresponds to an isotropic line joining an isotropic point outside from $H \cap X_1$ and a point $X_1 \cap X_j \cap H$ for some $j = 2, \ldots, 7$, which belongs an oval $\{X_1 \cap X_j \cap H \mid j = 2, \ldots, 7\}$ on the projective plane $H \cap X_1$. Thus the residue at X_1 is the dual of $T_2^*(O_4)$. By definition, the residue at a line is a generalized digon.

We may also verify that the above elements t_i $(i = 0, \ldots, 6)$, d and $e_i := M(0, \mathbf{0}; \mathbf{e}_i)$ $(i = 1, 2, 3)$ satisfy the presentation $\mathscr{R}' \cup \mathscr{F}' \cup \mathscr{A} \cup \mathscr{T}$ in 4.2 for the group G of type F_2, where $\mathbf{e}_i$ is the ith natural basis of $\mathbf{F}_4^3$. Furthermore, the full automorphism group of the above geometry $\mathscr{G}$ is given by adjoining to KA a semi-linear transformation of $\Gamma U_6(2)$ inducing the field automorphism on X_1. It might be interesting to note that there is a maximal subgroup of $U \simeq SU_6(2)$ isomorphic to $3M_{22}$ (non-split extension) containing $A \simeq 3A_7$.

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Received 24 January 1992 and accepted in revised form 15 July 1992

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